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FOREWORD TO THE SPECIAL "FCAA" ISSUE DEDICATED TO THE 40 YEARS OF CAPUTO DERIVATIVE AND TO THE 80th ANNIVERSARY OF PROF. CAPUTO

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It was *just 40 years ago*, in 1967, when Professor Michele Caputo published in *Geophysical Journal of the Royal Astronomical Society* a paper [2], where he provided via the Laplace transform a definition of a fractional derivative that, only later was recognized to be in many cases more suitable for applications to initial value problems of physical interest than the classical Riemann-Liouville derivative. In recent years, mainly due to the 1999 book by Podlubny [4], that definition was labeled as *Caputo derivative.* In his pioneering paper [2], Professor Caputo had introduced a rheological model that generalized the standard Voigt model by replacing the first time derivative with a derivative of order less than one, in order to better fit data of dissipation in the Earth (free oscillations and seismic surface waves) which exhibit a quality factor Q almost independent of frequency. This work was the second one with this aim since one year earlier he had published another paper [1], where the derivative of non-integer order was defined in a preliminary way via a formal fractional derivation of a power series for an analytic function. In the 1967 paper, Prof. Caputo had thus provided a rigorous definition via the Laplace transform, that however was completely neglected for many years by mathematicians, perhaps because they were not attracted by the physical applications.

In the following, as a former pupil of Professor Caputo, I would like to present some personal records of my interaction with him, started just when he arrived at the University of Bologna from North-America to cover the chair of Geodesy, and he published a book [3] (unfortunately only in Italian) on mathematical aspects of the Earth rheology, with the general title *"Elasticity and Dissipation"*. In this book he illustrated a number of applications to wave propagation and oscillation problems by using his innovative methods based on the derivative of non-integer order.

In those years Professor Caputo gave two courses at the Physics Department: one on Geodesy and one on Mathematical Methods of Physics. It was just due to his inclination towards Mathematical Physics that I, after getting a bachelor degree in Theoretical Physics of elementary particles (Quarks), was attracted by him and decided to become his graduate student in the framework of the "Scuola di Perfezionamento in Fisica" (a sort of PhD course in Physics, since PhD was established in Italy only about 15 years later!)... Having started to create a research group of undergraduate and graduate students in theoretical and experimental aspects of Geodesy and Seismology, Prof. Caputo offered to me some topics for research thesis including these on the applications of "his" derivative in the rheology of the Earth. I promptly decided to choose such an attractive and new topic and I worked to write a thesis under his supervision on rheological models characterized by constitutive relations containing derivatives of non-integer order.

Besides being inspired by the works of my supervisor [1]-[3], I had explored the literature of linear viscoelasticity and mathematics, and I had found very interesting the works by Gross [5]-[6] published in 1947, 1953 (for general viscoelasticity and for the Mittag-Leffler function as a creep and relaxation law) and the treatise by Gel'fand and Shilov [7], published in 1967 (for a proper definition of derivative of complex order in the framework of distributions). Two papers were published in 1971 on this respect, co-authored by M. Caputo and F. Mainardi. The former, [8] was published as an original note, in a journal of geophysical interest, *"Pure and Applied Geophysics"*, or simply "*Pageoph*", in which the Caputo model (or fractional Voigt model) was generalized to yield the so-called fractional Standard Linear Solid; the latter, [9] was published as a review paper in "*La Rivista del Nuovo Cimento*" ("*Il Nuovo Cimento*" is the official journal of the Italian Society of Physics (SIF) since former times), in which several aspects of linear viscoelasticity were examined. In particular (see Section 4, Table VI), a number of "fractional" models were considered as particular cases of the Fractional Standard Linear Solid (and therefore different from that introduced in *Pageoph*). Please, note that, in those times, both authors were NOT aware that other "forerunners" before them (including the most famous Scott-Blair) had proposed rheological constitutive equations based on fractional derivatives.

As I often like to say, Fractional Calculus is an old and yet novel topic, since it is from time to time re-discovered or used in an implicit way. This is the case of the rheological theory of Professor Rabotnov in the former Soviet Union on the use of singular Volterra operators, of which I became aware in October 1973. Then I attended at CISM (International Centre for Mechanical Sciences, Udine, North Italy, see http://www.cism.it) a course on Rheology where this eminent Soviet scientist was invited as a lecturer. Unfortunately, Acad. Rabotnov was not able to come (in those times Soviet scientists had difficulties to travel to Western countries). However, he provided CISM with typed lecture notes in English, which were distributed to the attendants. What a big surprise I had when, reading the lecture notes by Rabotnov, entitled "*On the Use of Singular Operators in the Theory of Viscoelasticity*" (pp. 39 + 3 figures), I became aware that the model proposed by the Russian scientist in 1948, [10] (note, after the 1947 Gross paper [5] of my knowledge) coincided with the Fractional Standard Linear Solid. The Russian author did not use the tools of the Fractional Calculus but he arrived for creep and relaxation laws to a special function (related to the Mittag-Leffler function), which was named after him as the Rabotnov function. Then, I started a kind correspondence by (ordinary) mail with the eminent scientist, who recognized the independent and similar work carried out in Bologna (Rabotnov replied in French). In October 1974 (from 24 to 29) another course on Rheology was organized by CISM in the framework of a session devoted to Experimental Methods in Mechanics. Professor Rabotnov was invited as a Lecturer for 8 hours, but again he was not able to come; as usual, he provided CISM with his lecture notes (of pp. 77 + 10 figures) entitled "*Experimental Evidence of the Principle of Heredity in Mechanics of Solids*". What a satisfaction I had in reading in Rabotnov's notes two explicit citations to the Caputo-Mainardi review paper in *Rivista del Nuovo Cimento*. Hereafter we report *these citations*:

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1. Pag. 45: . . . That's why it was of great interest for me to know the paper of Caputo and Mainardi /39/ from the University of Bologna published in 1971. These authors have obtained similar results independently without knowing the corresponding Russian publications. . . .

2. Pag. 47-48 (concerning the specific dissipation function): . . . Analogous formulae for dissipation are given in the above mentioned paper by Caputo and Mainardi /39/. The reasonings of these authors were slightly different. The derivative of an arbitrary order d^α/dt^α is determined as the inversion of the operator J^α; like the ordinary derivative of the order one can be considered as inversion of the operator of integration J^1. The derivatives of an arbitrary order were introduced first by Liouville, they have

been studied in an early work by Riemann. Like the ordinary differential equations, the differential equations containing derivatives of an arbitrary order can be studied. Caputo and Mainardi generalize the well known model of the standard viscoelastic body replacing the ordinary derivatives by the generalized ones of order ν. In this way they obtain the following equation:

$$\left(1 + a\frac{d^\nu}{dt^\nu}\right)\,\sigma(t) = \left(m + b\frac{d^\nu}{dt^\nu}\right)\,\epsilon(t)\,.$$

From this equation as particular cases the equations for the generalized Newton fluid, Voigt body and Maxwell body could be obtained selecting certain specialized values of constants. Following the path traced by Volterra we tried to avoid any kinds of differential equations, ordinary or generalized. The paper of Caputo and Mainardi contains a lot of experimental data of different authors in support of their theory. On the other hand a great number of experimental curves obtained by Postnikov and his coworkers as also by foreign authors can be found in numerous papers of Shermergor and Meshkov.

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I like to report the above sentences to point out the scientific honesty of Professor Rabotnov, who, however, did not cite (probably for reasons not depending on him) the works of Caputo and Mainardi in the revised version of his book [11]: *"Elements of Hereditary Solid Mechanics"*, English Translation, MIR Publ., Moscow, 1980, pp. 387 (revised from the 1977 Russian edition). In fact, therein no reference to the Caputo-Mainardi paper was present, in spite of the fact that he cited several papers by American scientists...

My interaction with Professor Caputo was lost just in those years, since he left Bologna to cover a more prestigious chair in Seismology at the University of Rome "La Sapienza". While Prof. Caputo continued to publish papers on the applications of "his" derivative up to nowadays (see the list of his publications), I moved to other fields of research (wave propagation and diffusion). This was also in view of the bad reaction and strong attacks that I received at conferences, when the majority of attendants considered "rubbish" the topic concerned with fractional derivatives to which no theoretical physical interpretation was attached. I limited myself to point out to Prof. Caputo that the issue of June 1984 of the magazine of the American Society of Mechanical Engineering: *J. Appl. Mech. (Trans. ASME)* published a series of two papers on the use of Fractional Calculus in Rheology. The former article [12] written by P.J. Torvik and R.L. Bagley was essentially based on the PhD thesis of Bagley under the supervision of Prof. Torvik (1977). The latter article [13] written by R.C. Koeller was based on the Volterra theory and henceforth on the book by Rabotnov (1980). In both works the authors cited our article [8] but not our review paper [9].

It was only in 1993 that I went back to the applications of Fractional Calculus when I became aware of the increasing literature (not always without errors or misunderstandings) on the application of fractional order operators inspired by the *fashion of Fractals*. The summer of 1994 was relevant for renewing my former acquaintance with Prof. Caputo's research activity in the theory and applications of fractional calculus. In fact, having organized together with Igor Podlubny, in the framework of the 14th World Congress on Computational and Applied Mathematics, Atlanta, Georgia (USA), a mini-symposium on the applications of Fractional Calculus, I invited him and he willingly accepted. This conference gave the opportunity to Podlubny to make personal acquaintance with Prof. Caputo and to recognize the prominent role of the Caputo derivative.

In a note [14], published in this same issue of *FCAA* dedicated to the 80-th birthday of Professor Caputo, Gorenflo and I have tried to briefly recall the main ideas about the Caputo derivative and its connection with the Riemann-Liouville derivative, in the framework of a tutorial survey where the differential equations of fractional order of relaxation and viscoelsticity are considered. Some historical notes are also presented there. The Caputo derivative is nowadays understood as a regularization of the classical fractional derivative introduced independently by Dzherbashyan and Nersesian [15] in 1968, therefore by some authors the Caputo derivative is referred to also as the Caputo-Dzherbashyan derivative.

I would like to conclude that in 1967 Prof. Caputo was 40 years old, so this year we have the pleasure to congratulate him also to *his 80-th birthday* (5 May 1927). By solicitation of the Managing Editor, Virginia Kiryakova, several papers in Volume **10** of *FCAA* are dedicated to this happy event. I like to recall that Professor Caputo is a Honorary Member of the Editorial Board of *FCAA* yet from the birth of the journal, that was in 1998.

I wish Professor Michele Caputo strong health and happiness. With all my respect to him,

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